Deformation Theory - Exercises

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Suggested reading:

- Szendroi - The unbearable lightness of deformation theory. https://people.maths.ox.ac.uk/szendroi/defth.pdf
- Huybrechts - Complex Geometry (Chapter 6).

1. Let $R = k[x_1, ..., x_n]$ be a polynomial ring and let $f \in R$. Define a first-order deformation of f to be an element of $R[t]/t^2$ which reduces to f modulo t. Say that such a deformation is trivial if it is related to f by an automorphism $\psi : R[t]/t^2 \to R[t]/t^2$ such that ψ is the identity modulo t. Prove that the space

$$\frac{\text{first-order deformations of } f}{\text{trivial deformations}}$$

is the Jacobi ring:

$$\mathrm{Jac}(f) = R \,/\, \langle \partial_1 f, ..., \partial_n f\rangle$$

2. Let R be an algebra over a field k. Let M be an R-module. Define a first-order deformation of M to be a module N over $R[t]/t^2$ such that (i) N is free as a $k[t]/t^2$ module, and (ii) $N/t = M$. Show that this is the same data as a short exact sequence of R-modules:

$$0 \to M \to N \to M \to 0$$

3. Cover $\mathbb{CP}^1$ by the two standard charts so the transition function is $\psi : x \mapsto 1/x$.

   (a) What are the possible defomations of ψ, using algebraic functions only, such that ψ remains an automorphism? Now show that they all give trivial deformations of $\mathbb{CP}^1$.

   (b) Repeat part (a) but for first-order deformations. *Hint: what are the automorphisms of* $\mathbb{C}[x, x^{-1}, t]/t^2$ *that reduce to* ψ *modulo* t*?*

4. Compute $H^1(\mathbb{CP}^n, T_{\mathbb{CP}^n})$ and deduce that $\mathbb{CP}^n$ has no first-order deformations. *The Euler sequence might be useful.*

5. On a real manifold M the space of differential forms $(\Omega^\bullet_M, d_{dR})$ is a dga. What deformation problem is governed by this dga? *Hint: what kind of thing is pertubed by adding a one-form?*

6. (a) Let L be an algebraic line bundle on an affine variety. Show that L has no non-trivial first-order deformations.

   (b) Let L be a line bundle on an affine elliptic curve. Is it true that L has no deformations? *Conclude that the moduli space of coherent sheaves on an affine variety is pretty weird.*

7. Let A be a finite-dimensional vector space over a field k. Let $A^\vee = \mathrm{Hom}_k(A, k)$ and let

$$B = TA^\vee = \bigoplus_{k\geq 0} (A^\vee)^{\otimes k}$$

be the free graded non-commutative algebra generated by $A^\vee$ in degree 1 (if $\dim A = n$ then B is an algebra of non-commutative polynomials in n variables, graded in the obvious way).

(a) Suppose we have a linear map $m : A \otimes A \to A$, hence $m^\vee : A^\vee \to (A^\vee)^{\otimes 2}$. This extends to a unique derivation:

$$Q : B \to B$$

Show that m is associative iff $Q^2 = 0$.

(b) Deduce the dgla that controls deformations of a non-commutative algebra (A, m).

8. (a) Let $R = k[x, y]$. Find a free resolution of the diagonal R-bimodule and hence compute the Hochschild cohomology of R. Deduce the possible first-order deformations of R as an associative algebra.

   (b) Repeat (a) for $\mathbb{A}^n$, *i.e.* $\mathbb{R} = k[x_1, ..., x_n]$. *You might need to learn about Koszul resolutions.*